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CONE-LENGTH AND LUSTERNIK–SCHNIRELMANN CATEGORY

OCTAVIAN CORNEA

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1. INTRODUCTION

FOR a topological space X, the Lusternik–Schnirelmann category, $catX$, is defined to be the least natural number n such that there are $n + 1$ open subsets of X which cover X and are (each) contractible in X.

This number was introduced in [10] where it was showed that, when X is a smooth manifold, $catX + 1$ gives a lower bound for the number of critical points of a smooth function on X. The definition we use in this paper for the L.S.-*cat* assigns to a point the category 0 and not 1 as it appears sometimes in the literature. It is easy to see that $catX$ is an invariant of the homotopy type of X.

From now on, unless otherwise specified, we will work inside the pointed category of spaces of the homotopy type of pointed CW-complexes.

For path-connected spaces of this kind there are two other equivalent definitions of *cat*.

The first one, introduced by Whitehead, asserts that $catX \leq n$ iff the diagonal $\Delta^{n+1}: X \to X^{n+1}$ factors through the fat wedge $T^{n+1}X \subseteq X^{n+1}$.

The second, belonging to Ganea, states that $catX \leq n$ iff the fibration $\Omega X^{*(n+1)} \xrightarrow{i_n} B_n\Omega X \to X$ has a section. Here $\Omega X^{*(n+1)}$ is the $(n + 1)$-fold join of ΩX and $E_n\Omega X = \Omega X^{*(n+1)} \xrightarrow{i_n} B_n\Omega X$ is the n-th filtration in Milnor's construction for the classifying space of ΩX.

Very often $B_n\Omega X$ is called the n-th space of Ganea associated to X, [3].

Remark. The L.S.-*cat* is an interesting notion for spaces more general than the ones we concentrate on here (see [8]). Following Dold, for a path-connected and well pointed space X we may define the category as $1+$ the least number of sets in an open, numerable covering of X with subsets containing $*$ and contractible in X. This definition is equivalent with both the pointed and unpointed version of Whitehead's definition.

For simply-connected rational spaces of finite type there is still another description of the category given by Felix and Halperin [4]. Let's first define a related notion (all our rational differential graded algebras (dga's) will be commutative).

Definition 1. Let $\mathscr{A}$ be a rational, augmented dga. The nil-length of $\mathscr{A}$, nil $\mathscr{A}$, is the least natural number n such that there is another rational, augmented dga $\mathscr{B}$, quasi-isomorphic to $\mathscr{A}$, such that $\bar{\mathscr{B}}$ is nilpotent of order $n + 1$. If no such n exists, then $nil\,\mathscr{A} = \infty$.

Felix and Halperin's description of the category asserts that a rational space X' has $catX' \leq n$ if a free simply connected dga of finite type, representing the homotopy type of

X', is a homotopy retract of a dga of $nil \leq n$. This description is based on the fact that the n-th Ganea space is represented by such a nilpotent dga.

For further use let $nilX' = nil(\mathcal{M}(X'))$ (where $\mathcal{M}(X')$ is Sullivan's minimal model of X', [12]). If X' is the rationalization of a simply connected CW-complex X, we may interpret $nilX'$ as the "homotopic nilpotency" of the algebra of P.L. forms on X.

The starting point for the present paper is a study by Lemaire and Sigrist [9] of the *cat* of rational spaces. They define, for a rational free dgL (differential graded Lie algebra) $\mathbf{L}$, its cone-length $Cl\mathbf{L}$ to be the least m such that there is a free dgL, $(\mathbf{L}', d)$, quasi-isomorphic to $\mathbf{L}$ and a filtration on $\mathbf{L}'$,

$$0 = F_0\mathbf{L}' \subseteq F_1\mathbf{L}' \subseteq \ldots \subseteq F_m\mathbf{L}' = \mathbf{L}'$$

such that $F_i\mathbf{L}' = \mathbf{L}(T_i)$, with $dT_i \subset F_{i-1}\mathbf{L}'$, $i \geq 1$ (if no such m exists, then, of course, $Cl\mathbf{L} = \infty$; $\mathbf{L}(T_i)$ is the free Lie algebra on T_i). The cone-length of X' is defined to be the cone-length of its Quillen model. Geometrically this means that X' has $ClX' \leq n$ iff there is a sequence of cofibration sequences:

$$H_i \to Y_i \to Y_{i+1}, \quad 0 < i < n$$

with H_i, Y_i rational spaces, Y_1 and H_i a bouquet of spheres for $i > 0$ and $Y_n \simeq X'$. In the same paper it was showed that $Cl = cat$ for formal and coformal spaces and it was conjectured that the equality holds for all rational spaces.

A few years later it was proved by Felix and Thomas, [5], that this conjecture is valid for spaces of category 2.

Remark. It is worth mentioning here that a relation between *cat* and a notion of cone-length appeared earlier in the work of Ganea. Indeed, let X be a path-connected space. Ganea, [6], introduced the notion of strong category of X: $CatX =$ the least natural number n for which there is a covering of a space with the homotopy type of X by $n + 1$ open, contractible subsets. He proved that $CatX \leq n$ iff there are cofibration sequences:

$$L_i \to X_i \to X_{i+1}, \quad 0 \leq i < n$$

with $X_0 \simeq *$ and $X_n \simeq X$. It is clear that $CatX \geq catX$. It was later proved by Ganea and Takens, [11], that $CatX \leq catX + 1$.

In this paper we introduce a new type of cone-length:

Definition 2. Let X be of the homotopy type of a path-connected, pointed CW-complex. The cone-length of X, $\mathrm{Cl}X$, is the least n for which there are cofibration sequences:

$$\Sigma^i Z_i \to X_i \to X_{i+1}, \quad 0 \leq i < n \tag{1}$$

with $X_0 \simeq *$ and $X_n \simeq X$ (for notational convenience the 0-suspension of a space is taken to be the space itself, $\Sigma^0 Z = Z$). If no such n exists, $\mathrm{Cl}X = \infty$.

Remarks. a. Initially we intended to use a "milder" cone-length in whose definition appears, for each $i > 0$, a single suspension ΣZ_i (instead of $\Sigma^i Z_i$). The notion obtained in this way will be denoted by clX. The definition of ClX above was suggested to us by Dieter Puppe.

b. It is obvious that $ClX \geq clX \geq CatX \geq catX$. Any co-H-space which is not a suspension is an example of space with $Cat = 2$ and $cat = 1$.

c. *A priori* we have, for rational spaces, two invariants denoted by ClX: one obtained using the rational variant of the above definition, and the other given by the original definition of Lemaire and Sigrist. In fact they agree because rational suspensions are wedges of spheres and, rationally, (1) means just that the $i+1$-level in a cone decomposition of X is obtained by attaching to the i-level cells of dimension at least $i+1$. But this happens in any reasonable "short" cone decomposition of X.

In the second section of the paper we will prove:

THEOREM 1.1 *For X as above, if* $\operatorname{cat} X = n$, *then there is a space Z such that* $\operatorname{Cl}(X \vee \Sigma^n Z) \leq n$.

Some interesting consequences are:

COROLLARY 1.2. $\operatorname{Cl} X \leq \operatorname{cat} X + 1$

COROLLARY 1.3. *If* $k > \operatorname{cat} X$, *then* $\operatorname{cat}(B_k \Omega X) = \operatorname{cat} X$.

Remarks. a. It is a wellknown fact that for $k \leq catX$, $cat(B_k\Omega X) = k$. That together with our result gives a precise description of $cat(B_k\Omega X)$ for all integers k.

b. The above result is stronger than a recent one of Dold's, obtained by different methods, which asserts that $k > catX$ implies $cat(B_k\Omega X) \leq catX + 1$.

c. Incidentally we obtain, if X is a manifold, that the minimal number of critical points of a smooth function on X is bounded from below by ClX.

d. Recently Doeraene, [2], showed that the notion of L.S.-category can be formulated in the more general context of a type of category similar to a Quillen closed model category, but satisfying a suplimentary, important, axiom:

If, in the commutative, cube diagram below, the lateral faces are (homotopy) pull-backs and the bottom face is a push-out, then the top is a homotopy push-out.

$$\begin{tikzcd}
D' \arrow[rr] \arrow[dr] \arrow[dd] & & C' \arrow[dr] \arrow[dd, dotted] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
D \arrow[rr, dotted] \arrow[dr] & & C \arrow[dr, dotted] & \\
& A \arrow[rr] & & B
\end{tikzcd}$$

He called this type of category a J-category. Then he showed that inside a J-category, Whitehead's and Ganea's definitions of the L.S.-*cat* coincide. It was also proved by Hess and Lemaire [7] that, in this context, the original definition of *cat* can be formulated and that it agrees with the other two ones.

Theorem 1.1 and its corollaries are valid inside a J-category. Even if we prefer not to use the language of J-categories, we will write the proofs in such a way that a translation into that language will be more or less automatic.

There is considerable interest for product formulas for *cat* and *Cat* ([11], [1]). We will prove here such a formula for the cone-length.

PROPOSITION 1.4. $\operatorname{Cl}(X \times Y) \leq \operatorname{Cl} X + \operatorname{Cl} Y$.

In the third section we will turn to the rational context.

First we will analyze the case of the spaces of $cat = 2$. Using the geometric considerations of the second section we present a new proof for the result of Felix and Thomas giving the equality $cat = Cl$ in this case. After another related construction concerning these spaces we go to the main result of the section:

THEOREM 1.5. *For any simply-connected rational space of finite type X' we have* $\mathrm{Cl}\,X' = \mathrm{nil}\,X'$.

Remarks. a. This equality was believed to hold for some time ([4], [9]).

b. It seems to be reasonable to expect that, in fact, $nilX' = CatX'$ (this is certainly true for formal, coformal, and L.S.-category 2 spaces).

c. As all the homotopic invariants (that we are studying here) of the rationalization of a space are smaller or equal than the corresponding invariants of the initial space, a trivial consequence of the theorem above is that the minimal number of critical points of a smooth function on a (simply-connected) manifold is bounded from below by the "homotopic nilpotency" of its algebra of P.L. forms. This suggests a more direct (and possibly geometric) relation between critical point theory and some form of nilpotency of the (P.L.) de Rham algebra of the manifold.

2. CONE-LENGTH. GEOMETRIC PROPERTIES

The proof of Theorem 1.1 is based on the following proposition (which has some interest in itself):

PROPOSITION 2.1. *Let $F \xrightarrow{i} E \to B$ be a fibration with B, E path connected. Then* $\mathrm{Cl}(E/F) \leq \mathrm{Cl}\,B$ *(here E/F is, up to homotopy, the cofibre of $F \xrightarrow{i} E$).*

Proof. In the following each commutativity, pull-back or push-out diagram is understood to be valid up to homotopy. We will use the following two facts:

LEMMA 2.2. *Suppose that, in the commutative, cube diagram below, all lateral faces are pull-backs and that the bottom face is a push-out.*

$$\begin{tikzcd}
D' \arrow[rr] \arrow[dr] \arrow[dd] & & C' \arrow[dr] \arrow[dd, dotted, no head] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
D \arrow[rr, dotted, no head] \arrow[dr] & & C \arrow[dr, dotted] & \\
& A \arrow[rr] & & B
\end{tikzcd}$$

Denote by F the common fibre of the vertical edges. The induced square:

$$\begin{tikzcd}
D'/F \arrow[r] \arrow[d] & C'/F \arrow[d] \\
A'/F \arrow[r] & B'/F
\end{tikzcd}$$

is a push-out.

Proof. The square

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & C' \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

is clearly a push-out and, in general, if we have a commutative diagram like:

$$\begin{tikzcd}
D'' \arrow[rr] \arrow[dr] \arrow[dd] & & C'' \arrow[dr] \arrow[dd, dotted] & \\
& A'' \arrow[rr] \arrow[dd] & & B'' \arrow[dd] \\
D' \arrow[rr, dotted] \arrow[dr] \arrow[dd] & & C' \arrow[dr, dotted] \arrow[dd, dotted] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
D''' \arrow[rr, dotted] \arrow[dr] & & C''' \arrow[dr, dotted] & \\
& A''' \arrow[rr] & & B'''
\end{tikzcd}$$

with the vertical edges being cofibration sequences and the top and middle level being push-out squares, then the bottom level is a push out diagram also. In our case $A'' = B'' = C'' = D'' = F$.

LEMMA 2.3.

$$((\Sigma^n X) \times Y)/(* \times Y) \simeq \Sigma^n((X \times Y)/(* \times Y))$$

Proof. In the commutative, cube diagram below the bottom is a push-out and the lateral faces are pull-backs.

$$\begin{tikzcd}
X \times Y \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd, dotted] & \\
& Y \arrow[rr] \arrow[dd] & & (\Sigma X) \times Y \arrow[dd] \\
X \arrow[rr, dotted] \arrow[dr] & & * \arrow[dr, dotted] & \\
& * \arrow[rr] & & \Sigma X
\end{tikzcd}$$

By applying the previous lemma we get a push-out square:

$$\begin{tikzcd} (X \times Y)/(* \times Y) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & (\Sigma X \times Y)/(* \times Y) \end{tikzcd}$$

That means $\Sigma((X \times Y)/(* \times Y)) \simeq (\Sigma X \times Y)/(* \times Y)$. This is the statement for $n = 1$. The cases $n > 1$ follow immediately by induction.

We return now to the proof of the proposition.

Suppose $ClB = n$. Then there are cofibration sequences

$$\Sigma^i Z_i \to X_i \to X_{i+1}, \quad 0 \le i < n$$

with $X_0 \simeq *$ and $X_n \simeq B$. Denote by $v_i\colon X_i \to B$ the obvious inclusions. Consider the pull-back fibrations induced by the v_i's:

$$F \to E_i \to X_i$$

We want to prove by induction that for $0 \le i \le n$, $Cl(E_i/F) \le i$. Clearly for $i = 0$ this is indeed the case as $E_0 = F$ and, hence, $E_0/F \simeq *$. Suppose now that $Cl(E_k/F) \le k < n$. We have the following commutative diagram

$$\begin{tikzcd}
(\Sigma^k Z_k) \times F \arrow[rr] \arrow[dr] \arrow[dd] & & F \arrow[dr] \arrow[dd, dotted] & \\
& E_k \arrow[rr] \arrow[dd] & & E_{k+1} \arrow[dd] \\
\Sigma^k Z_k \arrow[rr, dotted] \arrow[dr] & & * \arrow[dr, dotted] & \\
& X_k \arrow[rr] & & X_{k+1}
\end{tikzcd}$$

Where the bottom face is a push-out and the lateral faces are pull-backs. Apply now Lemma 2.2. We get a push-out square:

$$\begin{tikzcd}
(\Sigma^k Z_k \times F)/(* \times F) \arrow[r] \arrow[d] & * \arrow[d] \\
E_k/F \arrow[r] & E_{k+1}/F
\end{tikzcd}$$

But by Lemma 2.3 $(\Sigma^k Z_k \times F)/(* \times F) \simeq \Sigma^k((Z_k \times F)/(* \times F))$. This implies $Cl(E_{k+1}/F) \le k + 1$.

Remark. The previous construction gives a very explicit cone decomposition of E/F in terms of one for B.

We start now the proof of Theorem 1.1.

We have first to recall some of the properties of the spaces of Ganea $(B_n\Omega X)$. For a space X we have the fibrations

$$E_n\Omega X = (\Omega X)^{*(n+1)} \xrightarrow{i_n} B_n\Omega X \xrightarrow{t_n} X$$

The property that interests us here is that $B_{n+1}\Omega X$ is, up to homotopy, the cofibre of i_n and that the map t_{n+1} is induced in the obvious way (by sending $E_n\Omega X$ to a point) by t_n. Also, under the equivalence $B\Omega X \simeq X$, t_n corresponds just to the inclusion $B_n\Omega X \subseteq B\Omega X$. Clearly $B_0\Omega X \simeq *$. Recall that $A * B = \Sigma(A \wedge B)$, as a consequence of this:

$$(\Omega X)^{*(n+1)} = \Sigma^n(\wedge^{(n+1)}\Omega X) \tag{2}$$

Suppose that $cat\ X \le$ n. That means, by Ganea's description of cat, that $t_n\colon B_n\Omega X \to X$ admits a section s. This implies the existence of the commutative diagram below, whose rows and columns are fibrations:

$$\begin{tikzcd}
\Omega E_n \Omega X \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & E_n \Omega X \arrow[d] \\
\Omega E_n \Omega X \arrow[r, "j"] \arrow[d] & X \arrow[r] \arrow[d, "id"'] & B_n \Omega X \arrow[d] \\
* \arrow[r] & X \arrow[r, "id"] & X
\end{tikzcd}$$

It follows that j is homotopically trivial, hence $X/\Omega(E_n\Omega X) \simeq X \vee \Sigma\Omega(E_n\Omega X)$. Making use of the formula:

$$\Sigma\Omega\Sigma A \simeq \Sigma(\vee_{k\geq 1}(\wedge^{(k)} A)) \tag{3}$$

we get that $\Sigma\Omega(E_n\Omega X) \simeq \Sigma^n Z$ for some Z (which can be determined exactly from (2) and (3)).

Apply now the Proposition 2.1 to the fibration

$$\Omega E_n \Omega X \to X \to B_n \Omega X$$

This will give $Cl(X \vee \Sigma^n Z) \leq n$.

The proof of Corollary 1.2 is very easy now. Indeed X is the cofibre of the inclusion: $\Sigma^n Z \to X \vee \Sigma^n Z$.

Notice that the proof of the Theorem 1.1 gives a precise cone decomposition of $X \vee \Sigma^n Z$ and, consequently, of X. In this context it is useful to have:

Lemma 2.4. *Let* $F \to E \to B$ *be as before. Suppose* $\mathrm{Cl}(B \vee H) \leq n$, *then* $\mathrm{Cl}((E/F) \vee (H \times F)/(* \times F)) \leq n$.

Proof. Consider the fibration $F \to G \to B \vee H$ induced by the map $B \vee H \to B$ which collapses H to the base point. Clearly this fibration is trivial over H. Hence we have the following commutative diagram:

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr] \arrow[dd] & & H \times F \arrow[dr] \arrow[dd, dotted] & \\
& E \arrow[rr] \arrow[dd] & & G \arrow[dd] \\
* \arrow[rr, dotted] \arrow[dr] & & H \arrow[dr, dotted] & \\
& B \arrow[rr] & & B \vee H
\end{tikzcd}$$

The conditions needed for applying Lemma 2.2 are satisfied and this gives us the commutative push-out square:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & (H \times F)/(* \times F) \arrow[d] \\
E/F \arrow[r] & G/F
\end{tikzcd}$$

That means $G/F \simeq E/F \vee (H\times F)/(*\times F)$. By Proposition 2.1 we also have $Cl(G/F)\leq n$.

For the Corollary 1.3, remark that E/F is a retract of G/F, hence $cat(E/F)\leq cat(G/F)\leq n$. Using Theorem 1.1 we get $cat(E/F)\leq catB$ and we may apply this to the fibration $E_k\Omega X\to B_k\Omega X\to X$.

Remarks. a. The inequality $cat(E/F)\leq catB$ can also be obtained directly by using Whitehead's definition of *cat*. Using this definition $catX\leq n$ is equivalent to the existence of a covering of B with closed sets A_i, $0\leq i\leq n$, such that there are pointed homotopies $h^i\colon B\times[0,1]\to B$, $h^i_0=id$, $h^i_1(A_i)=*$. Lift h^i to $H^i\colon E\times[0,1]\to E$ such that $H^i_0=id$. Let $T_i\subset E$ be the preimage of A_i. Then $H^i_1(T_i)\subset F$ and $H^i_t(F)\subset F$ for all $t\in[0,1]$. We get induced homotopies $\bar{H}^i\colon E/F\times[0,1]\to E/F$ and A_i/F, $1\leq i\leq n$ will cover E/F, $\bar{H}^i_0=id$, $\bar{H}^i_1(A_i/F)=*$.

We will prove now the product formula (Proposition 1.4): $Cl(X\times Y)\leq ClX+ClY$.

Proof. Let $ClX=m$, $ClY=n$. There are cofibration sequences:

$$U_i\xrightarrow{f_i}A_i\longrightarrow A_{i+1},\quad V_j\xrightarrow{g_j}B_j\longrightarrow B_{j+1}$$

with $0\leq i<m$, $0\leq j<n$, $A_m\simeq X$, $B_n\simeq Y$, $U_i=\Sigma^iZ_i$, $V_j=\Sigma^jW_j$, $A_0=B_0=*$. We may, of course, assume $A_{i+1}=A_i\bigcup_{f_i}CU_i$, $B_{j+1}=B_j\bigcup_{g_j}CU_j$ (CX being the cone over X). This gives an increasing filtration for X and one for Y:

$$X=A_m\supset A_{m-1}\supset\ldots\supset A_0=*$$

$$Y=B_n\supset B_{n-1}\supset\ldots\supset B_0=*$$

such that $A_{i+1}-A_i=CU_i-U_i$, $B_{j+1}-B_j=CV_j-V_j$. Here U_i (resp. V_j) is identified with the "boundary" of the cone CU_i (resp. CV_j). Define $T_{ij}=(A_i-A_{i-1})\times(B_j-B_{j-1})$, $\overline{T_{ij}}$ the closure of T_{ij}, $D_{ij}=CU_{i-1}\times CV_{j-1}$, $S_k=\bigcup_{i+j=k}T_{ij}$, and $R_t=\bigcup_{k\leq t}S_k$. Here $0<i\leq m, 0<j\leq n, 0\leq k\leq m+n, 0\leq t\leq m+n$ and we consider A_k and B_k empty when $k<0$.

Notice that $X\times Y=R_{m+n}\supset R_{m+n-1}\supset\ldots\supset R_0=*$ is a filtration for $X\times Y$ and $R_t-R_{t-1}=S_t$, also $T_{ij}\cap T_{i'j'}$ is empty when $(i,j)\neq(i',j')$. Moreover if $i+j=t$, then $\overline{T_{ij}}-T_{ij}\subset R_{t-1}$.

Each D_{ij} is attached to R_{i+j-1} following a map

$$h_{ij}\colon U_i*V_j=(CU_{i-1}\times V_{j-1})\bigcup_{(U_{i-1}\times V_{j-1})}(U_{i-1}\times CV_{j-1})\to R_{i+j-1}$$

The map h_{ij} is obtained from the push-out square:

$$\begin{array}{ccc} U_{i-1}\times V_{j-1} & \xrightarrow{p_2} & V_{j-1}\\ \downarrow{\scriptstyle p_1} & & \downarrow\\ U_{i-1} & \longrightarrow & U_{i-1}*V_{j-1}\end{array}$$

(where p_1 and p_2 are projections) by $h_{ij}=u*v$ with

$$u\colon U_{i-1}\xrightarrow{f_{i-1}}A_{i-1}\longrightarrow A_{i-1}\times B_j\hookrightarrow R_{i+j-1}$$

$$v\colon V_{j-1}\xrightarrow{g_{j-1}}B_{j-1}\longrightarrow A_i\times B_{j-1}\hookrightarrow R_{i+j-1}$$

The previous remarks imply that we have a push-out diagram:

$$\begin{array}{ccc} \bigvee_{i+j=k} U_{i-1} * V_{j-1} & \longrightarrow & \bigvee_{i+j=k} D_{ij} \\ \downarrow & & \downarrow \\ R_{k-1} & \longrightarrow & R_k \end{array}$$

Recall that $D_{ij} \simeq *$ and $U_{i-1} * V_{j-1} = \Sigma^{i-1} Z_{i-1} * \Sigma^{j-1} W_{j-1} = \Sigma^{k-1}(Z_{i-1} \wedge W_{j-1})$ for $i + j = k$.

This gives a cofibration sequence: $\Sigma^{k-1}(Z_{i-1} \wedge W_{j-1}) \to R_{k-1} \to R_k$ for $0 < k \leq n + m$.

Remark. It is clear that the preceding results have analogues involving the "milder" notion of cone-length, clX. For this invariant the same type of arguments as those used in proving Theorem 1.1 lead to:

If X is a retract of Y, then there exists ΣZ such that $cl(X \vee \Sigma Z) \leq clY$ and, in particular, $clX \leq clY + 1$.

3. THE RATIONAL CASE

By contrast with the precedent section here we will pursue an algebraic study of the cone-length of rational spaces. All objects in this section will be rational and all spaces will be of finite type.

We start by making more precise the definition we will use. We will denote by $\mathbf{L}(X) = \mathbf{L}(x_1, \ldots, x_i, \ldots)$ the free Lie algebra on the set of generators $X = \{x_i\}_{i \in I}$.

Let $(\mathbf{L}, d) = (\mathbf{L}(X), d)$ be a free dgL and let $\mathscr{F} = \{F_k X\}$ be an increasing filtration on X. We will say it is a *differential* filtration if:

a. $0 = F_0 X, \ \cup F_k X = X$

b. $d(F_n X) \subset \mathbf{L}(F_{n-1} X)$

Clearly such a filtration induces one on $\mathbf{L}$ by $F_n \mathbf{L} = \mathbf{L}(F_n X)$. For $x \in \mathbf{L}$ denote by $fd(x)$ the filtration degree of x. Also denote by $F(\mathscr{F}, X)$ the length of $\mathscr{F}$ (this may also be infinite).

It is easy to see that $Cl\mathbf{L} = min\{F(\mathscr{F}', X') : (\mathbf{L}(X'), d') \simeq (\mathbf{L}, d)$ and $\mathscr{F}'$ is a differential filtration on $X'\}$.

Remark. As exemplified in [9] different systems of generators of the same dgL may admit differential filtrations of distinct length.

We begin by looking at some special results concerning the spaces of category 2.

A. If $catX = 2$ then $ClX = catX$. (This is the result of Felix and Thomas mentioned in the introduction.)

By Theorem 1.1 there is a bouquet of spheres $Z = \bigvee_i S^{t_i}$ with finitely many spheres in each dimension such that $Cl(X \vee Z) \leq 2$. Consequently it will be enough to know:

LEMMA 3.1. *If* $\mathrm{Cl}(X \vee S^n) = 2$, *then* $\mathrm{Cl}\, X = 2$.

Proof. Let $\mathbf{L} = (\mathbf{L}(G), d)$, $G = \{g_i\}_{i \in I}$ be a dgL of the homotopy type of a Quillen model for $X \vee S^n$ and $0 \subset G_1 \subset G_2 = G$ a length 2 differential filtration. We may choose G in such a way that $d(\sum u_i g_i) = 0$ with $u_i \in \mathbf{Q}$, $u_i \neq 0$ implies $g_i \in G_1$. Clearly we have $d(G_1) = 0$ and $d(G_2) \subset [G_1, G_1]$. We know that $H_*(\mathbf{L}(G), d) \simeq \pi_{*+1}(X \vee S^n)$ and $H_*(\mathbf{L}(G)/[\mathbf{L}(G), \mathbf{L}(G)]) \simeq H_{*+1}(X \vee S^n)$. If $h: \mathbf{L}(G) \to \mathbf{L}(G)/[\mathbf{L}(G), \mathbf{L}(G)]$ is the projection, then $h_*: H_*(\mathbf{L}(G), d) \to H_*(\mathbf{L}(G)/[\mathbf{L}(G), \mathbf{L}(G)])$ is identified with the Hurewicz homomorphism.

Denote by $\alpha \in \pi_n(X \vee S^n)$ the class of $S^n \hookrightarrow X \vee S^n$. Remark that $h_*(\alpha) \neq 0$. Choose $x \in \mathbf{L}$ of minimal filtration degree among the elements verifying $dx = 0$ and $h_*([x]) = h_*(\alpha)$.

Consider $\mathbf{L}_1 = (\mathbf{L} * \mathbf{L}(y), d_1)$, $d_1 = d$ on G, $d_1(y) = x$. Notice that L_1 represents the homotopy type of X. If $fd(x) = 1$, then define $fd(y) = 2$ and we get an obvious length 2 differential filtration on $G \cup \{y\}$.

Suppose $fd(x) = 2$. Write $x = g + m$ where $g = \sum a_i g_i$, $a_i \in \mathbf{Q}$, $m \in [\mathbf{L}, \mathbf{L}]$. If $fd(m) = 1$ we obtain $d(g) = 0$ which implies $fd(g) = 1$ and this contradicts the choice of x. Hence $fd(g) = 2 = fd(m)$. Let $\mathbf{L}_2 = \mathbf{L}_1/(y, dy)$, then, as the ideal (y, dy) is acyclic, $\mathbf{L}_1 \simeq \mathbf{L}_2$. It is easy to see that $\mathbf{L}_2$ is free. Indeed a way to think about $\mathbf{L}_2$ is to choose a $g_h \in G$ with $a_h \neq 0$ and replace it by g as a generator of $\mathbf{L}$. That is, let $G' = (G - \{g_h\}) \cup \{g, y\}$. Then $\mathbf{L}_2 \simeq \mathbf{L}(G' - \{g\})$ and the projection $p: \mathbf{L}_1 \to \mathbf{L}_2$ corresponds to a map sending $t \to t$ for $t \in G'$, $g \to -m$ and $y \to 0$. This map induces also a filtration on $\mathbf{L}_2$ by $fd(p(t)) = fd(t)$. It is clearly possible to choose the replaced g_h such that $fd(g_h) = 2$. In this case $fd(g_h) = fd(g) = fd(m) = 2$ and it is straightforward to check that this length 2 filtration is differential.

B. A special "short" model for spaces of category 2.

Recall the functor $\mathscr{C}: DGL \to DGCC_0$ where DGL is the category of dgL's and $DGCC_0$ is the category of differential, graded, cocomutative, connected coalgebras (dgc) [12]. For a dgl, $(\mathbf{L}, d)$, $\mathscr{C}(\mathbf{L})$ is, as a graded vector space, $\Lambda(s\mathbf{L})$ (s is the suspension; $\Lambda(X)$ is the free, graded, commutative, connected algebra on X). The coproduct is induced by the coalgebra structure of the tensorial coalgebra. On $\mathscr{C}(\mathbf{L})$ the differential δ is described by: $\delta = \delta_I + \delta_E$, $\delta_I(sx) = -sd(x)$, $\delta_E(sx_1 \wedge sx_2) = (-1)^{|x_1|} s[x_1, x_2]$.

Denote by $\{\mathscr{C}_i(\mathbf{L})\}$ the primitive filtration on $\mathscr{C}(\mathbf{L})$.

LEMMA 3.2. *If a finite type, connected dgL, $\mathbf{L}$, admits a differential filtration of length less or equal than 2, then there is a dgc $C \overset{i}{\hookrightarrow} \mathscr{C}(\mathbf{L})$ such that $C \subseteq \mathscr{C}_2(\mathbf{L})$ and i is a quasi-isomorphism.*

Proof. Again let $\mathbf{L} = (\mathbf{L}(G), d)$ and let the differential filtration be $0 \subset G_1 \subseteq G_2 = G$. If it is of length 1 assume $G_2 = G_1$. Remark that in this case, as d is trivial, if we take $\mathbf{Q}(G)$ as a dgc with trivial differential and coproduct, the inclusion $\mathbf{Q}(G) \hookrightarrow \mathscr{C}(\mathbf{L})$ is a quasi-isomorphism.

Suppose now the length of the filtration to be 2. We can assume that d is decomposable ($dG \subset [\mathbf{L}, \mathbf{L}]$). Indeed $d(G_1) = 0$ and if $g \in G_2 - G_1$, then $dg = \sum a_i g_i + m$ with $g_i \in G$, $m \in [\mathbf{L}, \mathbf{L}]$ and $fd(g_i) = fd(m) = 1$. If some $a_s \neq 0$, by a standard method we may eliminate g and g_s from $\mathbf{L}$ and it is easy to see that on the resulting dgL we can define (as in the proof of the previous Lemma) a length 2 differential filtration. Because of this assumption each of the elements of G will represent a homology class of $\mathscr{C}(\mathbf{L})$ and, also, each such homology class is represented by an element of G. Denote by $|x|$ the degree of $x \in \mathbf{L}$, we will indicate this degree by an upper index.

Suppose that for $k \leq n$ we have proved that $\mathbf{L}_k = \mathbf{L}(G_1 \cup (G_2)^{\leq k}, d)$ satisfies our claim. That is, there exists $C_k \subset \mathscr{C}_2(\mathbf{L}_k)$ and the inclusion $C_k \subset \mathscr{C}(\mathbf{L}_k)$ is a quasi-isomorphism. We will show that for any $g \in G_2 - G_1$, $|g| = k + 1$, there is a dgc, $C'' \subset \mathscr{C}_2((\mathbf{L}_k * \mathbf{L}(g), d))$, containing both C_k and sg and such that the inclusion $C'' \subset \mathscr{C}((\mathbf{L}_k * \mathbf{L}(g), d))$ is a quasi-isomorphism.

The construction of C'' goes as follows:

Let $dg = m$. We want first to construct a dgc C' such that: $C' \subsetneq \mathscr{C}(\mathbf{L}_k)$ induces a quasi-isomorphism, $sm \in C'$, $C_k \subset C'$ and $C' \subset \mathscr{C}_2(\mathbf{L}_k)$. If $sm \in C_k$, then $C' = C_k$.

If $sm \notin C_k$, express $m = \sum u_i[a_i, b_i]$, $a_i \in \mathbf{L}$, $b_i \in \mathbf{L}$, $u_i \in \mathbf{Q}$. We know $fd(a_i) = fd(b_i) = 1$. Let J be such that $s[a_i, b_i] \notin C_k$ iff $i \in J$. Take $C'_1 = C_k \oplus \bigoplus_{i \in J} (\mathbf{Q}(s[a_i, b_i]))$. But $s[a_i, b_i]$ represents a nontrivial homology class in C'_1. In fact, the $s[a_i, b_i]$'s, $i \in J$, will represent the kernel of $H_*(C'_1 \subsetneq \mathscr{C}(\mathbf{L}_k))$. Indeed, in $\mathscr{C}(\mathbf{L})$, $s[a_i, b_i] = \delta((sa_i) \wedge (sb_i))$. Consequently, we have to add to C'_1 the elements $(sa_i) \wedge (sb_i)$. But for preserving the coalgebra structure we must also have the sa_i's and sb_i's (for $i \in J$). So let $C'_2 = C'_1 + \sum_{i \in J} \mathbf{Q}(sa_i \wedge sb_i) + \sum_{i \in J} \mathbf{Q}(sa_i) + \sum_{i \in J} \mathbf{Q}(sb_i)$.

Remark that the possible kernel of $H_*(C'_2 \subsetneq \mathscr{C}(\mathbf{L}_k))$ appears in degree at most $max\{|a_i|, |b_i| : i \in J\} < |[a_i, b_i]| = |m|$. That means that our procedure (applied this time to sm) moves the kernel of the inclusion $C'_* \subset \mathscr{C}(\mathbf{L}_k)$ downwards (with respect to $|\cdot|$). Using it repeatedly, because of the finite type conditions, we will get C' with the required properties.

Take $C'' = C' \oplus \mathbf{Q}(sg)$ with $d(sg) = sm$.

It is clear that the construction applied to g can now be applied to the other elements of degree $k + 1$. Consequently, our claim will follow by induction.

We start now proving the Theorem 1.5.

First the inequality $ClX \leq nilX$.

Recall the functor $\mathscr{L}_*$: $\mathscr{L}_* = \mathscr{L} \circ \#$ where $\mathscr{L}: DGCC_0 \to DGL$ applied to a dgc, (C, δ), gives $\mathscr{L}(C) = (\mathbf{L}(s^{-1}C), d)$ with $d = d_1 + d_2$, $d_1(s^{-1}c) = -s^{-1}(\delta c)$, $d_2(s^{-1}c) = -1/2 \sum (-1)^{|c'_i|}[s^{-1}c'_i, s^{-1}c''_i]$ where $c \in C$, $\bar{\Delta}c = \sum c'_i \otimes c''_i$ [12]; $\#$ is a dualization functor $\#: DGA_0 \to DGCC_0$.

The idea for the proof comes from a simple remark: Let $(\Lambda X, d)$ be a free dga with d decomposable. Choose a vector space basis, B, for ΛX which consists of monomials and take all duals with respect to this basis. Suppose that for $b \in B$ we define $fd(s^{-1}b^*) =$ length of b as a monomial. It is easy to check that this filtration is differential.

The outline of the proof is the following:

1. Look for a certain type of filtration on a free Lie algebra of finite type which can be easily transformed into a differential one without increasing its length.
2. Show that for any finite type connected dga $\mathscr{A}$, with $\bar{\mathscr{A}}^{(n+1)} = 0$, we can get on $\mathscr{L}_*(\mathscr{A})$ a filtration like the one discussed at the previous point with length $\leq n$.
3. Show that if $nilX \leq n$, then X can be represented by a finite type, simply connected dga $\mathscr{A}'$ with $\bar{\mathscr{A}}'^{(n+1)} = 0$.

In this case $\mathscr{L}_*(\mathscr{A}')$ is a Quillen model of X and the previous two steps give a differential filtration of length less than n on $\mathscr{L}_*(\mathscr{A}')$.

PROPOSITION 3.3. *Suppose* $\mathbf{L} = \mathbf{L}(X)$, $X = \{x_i\}_{i \in I}$ *of finite type. If* $\mathscr{F} = \{F_*X\}$ *is an increasing filtration on* X *with the properties*:

a. $0 = F_0X$, $\cup F_iX = X$

b. *For each* x_i *write* $dx_i = \sum a_i^h x_h + m_i$, *where* $i, h \in I$ *and* $m_i \in [\mathbf{L}, \mathbf{L}]$.

Assume:

$$fd(x_i) > fd(m_i) \tag{4}$$

Then $\mathrm{Cl}\,\mathbf{L} \leq F(\mathscr{F}, \mathbf{L})$.

Proof. It is clearly enough to find a free dgL, $\mathbf{L}'$, and a quasi-isomorphism $f: \mathbf{L} \to \mathbf{L}'$ such that $f(X)$ contains a free system of generators of $\mathbf{L}'$ and with the property that if we define

$$fd(f(x_i)) = fd(x_i) \tag{5}$$

then the filtration induced on $\mathbf{L}'$ is differential.

The dgL $\mathbf{L}'$ (together with the morphism f) is obtained by iterating the following procedure.

Suppose that for each x_i with $i \in I$ and $|x_i| < k$ we have $fd(x_i) > fd(dx_i)$ and suppose there is an $x \in X$ which verifies $|x| = k$ and $fd(x) \leq fd(dx)$ (if for all k no such x exists we are done).

We will construct a dgL, $\mathbf{L}_1$, together with a quasi-isomorphism $f_1 : \mathbf{L} \to \mathbf{L}_1$ such that:

1. The set $f_1(X)$ contains a free system of generators of $\mathbf{L}_1$, Y.
2. The filtration induced on $\mathbf{L}_1$ by (5) satisfies conditions a. and b.
3. For some x_α with $fd(x_\alpha) \leq fd(dx_\alpha)$, $|x_\alpha| = k$, we have $f_1(x_\alpha) = 0$ and for $y \in Y$, $|y| < k$, $fd(y) > fd(dy)$.

Consider the set $A = \{i \in I : |x_i| = k, fd(x_i) \leq fd(dx_i)\}$. Recall that for each $u \in I$ we have $dx_u = \sum a_u^h x_h + m_u$. Let $\alpha \in A$ be such that $fd(m_\alpha) = min\{fd(m_j) : j \in A\}$. Select also a β such that $fd(x_\beta) = max\{fd(x_v) : a_\alpha^v \neq 0\}$. Let $\mathbf{L}_1 = \mathbf{L}/(x_\alpha, dx_\alpha)$.

Let $f_1 : \mathbf{L} \to \mathbf{L}_1$ be the projection. Denote $y_i = f_1(x_i)$ for $i \in I' = I - \{\alpha, \beta\}$. Notice that, as the ideal (x_α, dx_α) is acyclic, f_1 is a surjective quasi-isomorphism; it also satisfies:

1. $f_1(x_\beta) = (-f_1(m_\alpha) - \sum_{h \neq \alpha} a_\alpha^h y_h)/a_\alpha^\beta$.
2. $f_1(x_\alpha) = 0$

Clearly $Y = \{y_i\}$ is a free system of generators for $\mathbf{L}_1$.

Denote $m_i' = f_1(m_i)$. Let $i \in I'$. We have $dy_i = \sum_{h \in I'} a_i^h y_h + m_i' + a_i^\beta(-m_\alpha' - \sum_{h \neq \alpha} a_\alpha^h y_h)/a_\alpha^\beta$. Define now $fd(y_i) = fd(x_i)$.

We have to check if b. is satisfied. First notice $fd(x_\beta) \geq fd(m_\alpha + \sum_{h \neq \alpha} a_\alpha^h x_h)$ because $fd(dx_\alpha) \geq fd(x_\alpha) \geq fd(m_\alpha)$ and because of the choice of β. Consequently, $fd(f_1(x_\beta)) \leq fd(x_\beta)$. This implies $fd(m_i') \leq fd(m_i)$ for all $i \in I'$. Thus for all such i's we have $fd(y_i) \geq fd(m_i')$. This proves b. when $a_i^\beta = 0$. When $a_i^\beta \neq 0$ we need also $fd(y_i) \geq fd(m_\alpha')$. Clearly $|x_i| = k$. There are two possibilities: first $fd(x_i) > fd(dx_i)$; if this happens we have $fd(y_i) = fd(x_i) > fd(dx_i) \geq fd(x_\beta)$ (because $a_i^\beta \neq 0$) $\geq fd(m_\alpha) = fd(m_\alpha')$. We are left with the case $fd(x_i) \leq fd(dx_i)$. But, by the choice of α, we have $fd(m_i') = fd(m_i) \geq fd(m_\alpha)$.

Now we will see what kind of filtration we need on a dga $\mathscr{A}$ such that on $\mathscr{L}_*(\mathscr{A})$ we have a filtration of the type discussed in the previous proposition.

LEMMA 3.4. *Let $\mathscr{A}$ be a finite type, connected dga and let $B = \{g_i\}_{i \in I}$ be a basis (as a linear space) for $\mathscr{A}$ containing $1 \in \mathbf{Q}$. Suppose there is an increasing filtration on B, $\{B_i\}$, $0 \leq i \leq n$, such that $B_0 = 1$, $B_n = B$ and for each $g_i \in B$ we have*

$$fd(g_i) > max\{fd(g_h), fd(g_k)\} \tag{6}$$

whenever $u^i \neq 0$ in $g_h g_k = \sum u^s g_s$, where both g_h and g_k are different from 1. Then on $\mathscr{L}_*(\mathscr{A})$ we can find a filtration with the properties in the previous proposition and having length n.

Proof. We dualize everything with respect to B. Denote by $x_i = s^{-1}(g_i^*)$ for $g_i \neq 1$. The set $X = \{x_i\}_{i \in I}$ is a set of free generators for $\mathscr{L}_*(\mathscr{A})$. Define $fd(x_i) = fd(g_i)$. Checking that this is the right type of filtration is straightforward (of course, we add 0 as the 0-filtration).

The second step is concluded by:

LEMMA 3.5. *Let $\mathscr{A}$ be a finite type, connected dga. If $\bar{\mathscr{A}}^{(t+1)} = 0$, then there is a basis and a filtration of length at most t with the properties in the previous lemma.*

Proof. Take $\{x_1, \ldots x_l, \ldots\}$ a minimal system of generators for $\mathscr{A}$ as an algebra. Denote by $\mathscr{A}^n$ the elements of degree n in $\mathscr{A}$. The construction of our basis will be done by induction. Define $fd(1) = 0$.

Suppose that for $\mathscr{A}^t$ with $t < n$ we have constructed a basis consisting of monomials and a filtration satisfying the condition in the previous lemma and such that for the elements of this basis, b_i^t, $i \in I_t$, $t < n$, we have $fd(b_i^t) =$ length of b_i^t as a monomial in the x_i's. When we speak about the length of a monomial $b \in \mathscr{A}$ we understand the longest expression of b written as a monomial in the x_i's. Notice that, because of the finite type assumption, there are just finitely many such expressions for b, moreover, because $\bar{\mathscr{A}}^{(t+1)} = 0$, all such expressions are shorter or equal than t. In this way the length is well defined (otherwise it may not be because $\mathscr{A}$ is not free).

Now $\mathscr{A}^n$ is generated (as a linear space) by monomials of the type $m = \Pi x_i^{h_i}$, $h_i \in \mathbf{N}$ and $\sum h_i |x_i| = n$. Choose the longest such monomial, that is the one for which $l(m) = \sum h_i$ is maximal. Denote by b_1^n this monomial and define $fd(b_1^n) = l(b_1^n)$. It is clear that the condition (6) is satisfied in the case of b_1^n. Now suppose we have constructed $b_1^n, \ldots, b_k^n$ linearly independent elements in $\mathscr{A}^n$, monomials, such that $fd(b_i^n) = l(b_i^n) \geq l(b_{i+1}^n) = fd(b_{i+1}^n)$ and such that condition (6) is satisfied for each $i \leq k$.

Take b_{k+1}^n to be the longest monomial linearly independent with respect to $\{b^n_{\leq k}\}$. Define $fd(b_{k+1}^n) = l(b_{k+1}^n)$.

Let's check our critical condition. Take b_u^s, b_v^m with $s + m = n$. Express $b_u^s b_v^m = \sum_{i=1}^{k+1} h^i b_i^n + g$ where g is linearly independent with respect to $\{b^n_{\leq k+1}\}$, $h^{k+1} \neq 0$. We have $l(b_u^s b_v^m) > max\{l(b_u^s), l(b_v^m)\} = max\{fd(b_u^s), fd(b_v^m)\}$. Thus if $fd(b_{k+1}^n) = l(b_{k+1}^n) \geq l(b_u^s b_v^m)$ we have nothing more to prove. If this is not the case, because of the choice of b_{k+1}^n, that means $b_u^s b_v^m$ is linearly dependent with respect to $(b_1^n, \ldots, b_k^n)$. But that leads to contradiction because it implies that $(b_1^n, \ldots, b_k^n, b_{k+1}^n)$ are linearly dependent.

The third step is given by the following construction.

PROPOSITION 3.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be two dga's, $\mathscr{A}$ of finite type and simply-connected and $\mathscr{B}$ augmented such that there is a morphism $f\colon \mathscr{A} \to \mathscr{B}$ of augmented algebras inducing an isomorphism in cohomology. If $\bar{\mathscr{B}}^{(n+1)} = 0$, $n > 0$, then there exists a simply-connected finite type dga $\mathscr{A}'$, $\bar{\mathscr{A}}'^{(n+1)} = 0$, and a factorization of f:*

$$\mathscr{A} \xrightarrow{g} \mathscr{A}' \xrightarrow{f'} \mathscr{B}$$

with g being a quasi-isomorphism.

Proof. Clearly f factors $\mathscr{A} \xrightarrow{g} \mathscr{A}/\bar{\mathscr{A}}^{(n+1)} \xrightarrow{h} \mathscr{B}$. If $n = 1$, then we have an algebra isomorphism $\bigoplus_i \mathbf{Q}x_i \simeq \mathscr{A}/\bar{\mathscr{A}}^2$ for a certain set of x_i's; the sum is finite in each dimension. Let $K \subseteq \oplus \mathbf{Q}_i$ be the ideal generated by a system of representatives for $KerH^*(h)$. Obviously, we may consider $\mathscr{A}/\bar{\mathscr{A}}^2 = K \oplus ((\mathscr{A}/\bar{\mathscr{A}}^2)/K)$. It follows that we get a map $\mathscr{A} \to (\mathscr{A}/\bar{\mathscr{A}}^2)/K \to \mathscr{B}$ which is a quasi-isomorphism.

For the case when $n > 1$ we will use the next lemma.

LEMMA 3.7. *Let G be a connected, finite type dga. Suppose $f\colon G \to H$ is a quasi-isomorphism which factors $G \xrightarrow{p} G/J \xrightarrow{f'} H$ where J is an ideal of G. There exists a twisted tensor product $G/J \otimes_d \Lambda Y$ and maps such that the diagram below is commutative:*

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[dr, "f"'] & G/J \arrow[r, "h"] \arrow[d, "f'"] & G/J \otimes_d \Lambda Y \arrow[dl, "f''"] \\ & H & \end{tikzcd}$$

with $h \circ p, f''$ quasi-isomorphisms. Moreover the connectivity of Y is at least that of $J-2$ and Y is of finite type.

Proof. We construct ΛY by induction. First let $Y_0 = 0$ and $f_0 = f'$. Clearly $H^0(f_0)$ is an isomorphism and $H^*(f_0)$ is epi for $* \geq 0$. Suppose $f_i: G/J \otimes_d \Lambda Y \to H$ constructed such that $H^*(f_i)$ is an epimorphism for all $*$ and it is an isomorphism for $j \leq i$. Let $\{x_i\}$, $x_i \in (G/J \otimes_d \Lambda Y_i)^{i+1}$ be a system of representatives for a basis of $KerH^{i+1}(f_i)$. Let $Y_{i+1} = Y_i \oplus \{s^{-1}x_i\}$. Take on $G/J \otimes_d \Lambda Y_{i+1}$ the differential which extends the differential on $G/J \otimes_d \Lambda Y_i$ by $d(s^{-1}x_i) = x_i$. Extend f_i to $f_{i+1}: G/J \otimes_d \Lambda Y_{i+1} \to H$ by $f_{i+1}(s^{-1}x_i) = t_i$, where $t_i \in H$ verifies $dt_i = f_i(x_i)$. Passing to the limit we get our claim. Remark also that the connectivity of Y is one less the connectivity of f' and this one is at least the connectivity of J-1. As $KerH^*(f_i)$ is finite dimensional for each $*$, Y is of finite type.

Return now to the proposition. We apply repeatedly the lemma. We start with $\mathscr{A} \xrightarrow{g} \mathscr{A}/\bar{\mathscr{A}}^{(n+1)} \xrightarrow{h} \mathscr{B}$. Let $\mathscr{A}_1 = \mathscr{A}/\bar{\mathscr{A}}^{(n+1)} \otimes_d \Lambda Y^1$ given by the lemma together with the quasi-isomorphisms $\mathscr{A} \xrightarrow{t_1} \mathscr{A}_1 \xrightarrow{f_1} \mathscr{B}$. Now f_1 factors again $\mathscr{A}_1 \xrightarrow{g_1} \mathscr{A}_1/\bar{\mathscr{A}}_1^{(n+1)} \xrightarrow{h_1} \mathscr{B}$ and we may apply again the lemma. At the i-th step we get $\mathscr{A} \xrightarrow{t_i} \mathscr{A}_i \xrightarrow{f_i} \mathscr{B}$ with t_i and f_i quasi-isomorphisms and $\mathscr{A}_i = \mathscr{A}_{i-1} \otimes_d \Lambda Y^i$. Clearly, $A' = \lim \mathscr{A}_i$ is nilpotent of the right order and we have a diagram like in the statement. To conclude, we just have to see that $\mathscr{A}'$ is of finite type. For that it is enough to show that the connectivity of the Y^i's is strictly increasing. By the lemma the connectivity of Y^i is at least that of $\bar{\mathscr{A}}_{i-1}^{(n+1)} - 2$. But $\mathscr{A}_{i-1} = \mathscr{A}_{i-2} \otimes_d \Lambda Y^{i-1}$. Hence the connectivity of $\bar{\mathscr{A}}_{i-1}^{(n+1)}$ is at least 4 + that of Y^{i-1}, $(n > 1)$.

Remarks. a. If $\mathscr{A}$ in the previous proposition is the minimal model of a simply-connected, finite type space X we get a series of "obstructions" in order for X to be of $nil \leq n$. The first step is to ask for the projection $g_0 = g: \mathscr{A} \to \mathscr{A}/\bar{\mathscr{A}}^{(n+1)}$ to admit a retract up to homotopy (this is equivalent to $catX \leq n$). To formulate the $m+1$ obstruction suppose $g_m: \mathscr{A} \to \mathscr{A}_m/\bar{\mathscr{A}}_m^{(n+1)}$ constructed and suppose it admits a retract up to homotopy. Then we may construct $\mathscr{A}_{m+1}$ and a map $g_{m+1}: \mathscr{A} \to \mathscr{A}_{m+1}/\bar{\mathscr{A}}_{m+1}^{(n+1)}$. The $m+1$ obstruction requires for g_{m+1} to admit a retract up to homotopy.

b. The inequality $nil \geq Cl$ together with the next lemma gives another proof (in the rational context) of the fact that $Cl \leq cat + 1$.

LEMMA 3.8. *If $F \to E \to B$ is a fibration of finite type, simply-connected spaces, then* $\mathrm{nil}(E/F) \leq \mathrm{nil}\, B$.

Proof. Suppose $\mathscr{M}(B) \to \mathscr{B}$ is a quasi-isomorphism with $\mathscr{B}$ simply-connected of finite type and $\bar{\mathscr{B}}^{(n+1)} = 0$. We may represent the inclusion $F \subsetneq E$ by a projection $p: \mathscr{B} \otimes_d \mathscr{F} \to \mathscr{F}$ where $\mathscr{F}$ is a minimal model for F. Clearly $Kerp \oplus \mathbf{Q}$ represents E/F; on the other hand $Kerp = \bar{\mathscr{B}} \otimes_d \mathscr{F}$, hence it is nilpotent of the wanted order.

Applying this to the fibration $\Omega E_n(\Omega X) \to X \to B_n(\Omega X)$, when $n = catX$, we get that $nil(X \vee \Sigma\Omega E_n(\Omega X)) \leq n$. Along with $Cl \leq nil$, this implies $cat + 1 \geq Cl$.

We pass now to the inequality $nil \leq Cl$. The main part in proving it consists in showing that we can model a map $\Sigma Z \to X$ in term of "short" models.

PROPOSITION 3.9. *Let $\mathscr{A}$ be a free, simply-connected dga of finite type and let $f: \mathscr{A} \to \mathscr{A}_1$ be a quasi-isomorphism such that $\mathscr{A}_1$ is simply-connected of finite type and $\bar{\mathscr{A}}_1^{(n+1)} = 0$. Let*

$\mathscr{B}$ be an augmented dga with $\bar{\mathscr{B}}^2 = 0$. If $g\colon \mathscr{A} \to \mathscr{B}$ is a morphism of augmented dga's, then there exists a dga $\mathscr{A}_2$ of finite type, simply-connected with $\bar{\mathscr{A}}_2^{(n+2)} = 0$ and maps making the next diagram commutative:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "h"] \arrow[dr, "g"'] & \mathscr{A}_2 \arrow[r, "e"] \arrow[d, "\bar{g}"] & \mathscr{A}_1 \\ & \mathscr{B} & \end{tikzcd}$$

where both h and e are quasi-isomorphisms and $e \circ h = f$.

Proof. Write $\mathscr{A} = \mathscr{A}_0 \otimes \mathscr{F}$ with $\mathscr{A}_0$ minimal and $\mathscr{F}$ free acyclic. If $\mathscr{A}_0 = (\Lambda\{x_i\}, d)$ let $\mathscr{A}' = (\Lambda\{s^{-1}x_i\}, 0)$. By the standard construction of a K.S. model for the multiplication $\mu\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ ([3]) we get a twisted tensor product $T = (\mathscr{A} \otimes \mathscr{A}) \otimes_d \mathscr{A}'$ which fits into a commutative diagram:

$$\begin{tikzcd} \mathscr{A} \otimes \mathscr{A} \arrow[r] \arrow[dr, "\mu"'] & (\mathscr{A} \otimes \mathscr{A}) \otimes_d \mathscr{A}' \arrow[d, "\psi"] \\ & \mathscr{A} \end{tikzcd}$$

where ψ is a quasi-isomorphism. Clearly, we have inclusions $i_1\colon \mathscr{A} \to T$, $i_1(x) = x \otimes 1 \otimes 1$, $i_2\colon \mathscr{A} \to T$, $i_2(x) = 1 \otimes x \otimes 1$; both of them are quasi-isomorphisms. Recall also $d(s^{-1}x_i) = x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1 \, mod(\overline{\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}'^2})$. Let p be the projection $(\mathscr{A} \otimes \mathscr{A}) \otimes_d \mathscr{A}' \to \mathscr{A} \otimes_d \mathscr{A}'$ which sends $1 \otimes x \otimes 1$ to zero for $x \in \mathscr{A}$.

The inclusion i_1 makes T into an $\mathscr{A}$ module. Let $K = \mathscr{A}_1 \otimes_{\mathscr{A}} T = (\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}'$. Let $p_1 = \mathscr{A}_1 \otimes p$, $\psi_1 = \mathscr{A}_1 \otimes \psi$. Remark that $\psi_1\colon (\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}' \to \mathscr{A}_1$ is a quasi-isomorphism. The morphism g makes $\mathscr{B}$ into an $\mathscr{A}$ module. Consider K as an $\mathscr{A}$ module via the inclusion i_2' given by the composition of the map $T \to K$ with i_2. Let $S = \mathscr{B} \otimes_{\mathscr{A}} K = (\mathscr{A}_1 \otimes \mathscr{B}) \otimes_d \mathscr{A}'$. Denote $p_1' = \mathscr{B} \otimes p_1$.

We get the following diagram:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "g"] \arrow[d, "i_2'"'] & \mathscr{B} \arrow[d, "i_2''"] \\ K \arrow[r, "g'"] \arrow[d, "\psi_1"'] & S \\ \mathscr{A}_1 & \end{tikzcd}$$

Here both i_1' and i_2'' are quasi-isomorphisms, $g'\colon K = (\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}' \to (\mathscr{A}_1 \otimes \mathscr{B}) \otimes_d \mathscr{A}' = S$ is the obvious map and $\psi_1 \circ i_2' = f$.

We also have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & (\mathscr{A}_1 \otimes \bar{\mathscr{A}}) \otimes_d \mathscr{A}' \arrow[r, "j"] \arrow[d, "g''"'] & (\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}' \arrow[r, "p_1"] \arrow[d, "g'"'] & \mathscr{A}_1 \otimes_d \mathscr{A}' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (\mathscr{A}_1 \otimes \bar{\mathscr{B}}) \otimes_d \mathscr{A}' \arrow[r, "j'"] & (\mathscr{A}_1 \otimes \mathscr{B}) \otimes_d \mathscr{A}' \arrow[r, "p_1'"] & \mathscr{A}_1 \otimes_d \mathscr{A}' \arrow[r] & 0 \end{tikzcd}$$

where both short sequences are exact. It is easy to see that $\mathscr{A}_1 \otimes_d \mathscr{A}'$ is acyclic, hence both j and j' induce isomorphisms in cohomology in positive degrees. Let $\mathscr{B}_1 = Ker(j') \oplus \mathbf{Q}$, $R = Ker(j) \oplus \mathbf{Q}$. As $p_1 \circ i_2' = 0$, $p_1' \circ i_2'' = 0$, i_2' and i_2'' factor respectively through R and $\mathscr{B}_1$.

Notice now that both $\mathscr{B}$ and $\mathscr{B}_1$ are algebras with trivial multiplication. The inclusion $h_1 : \mathscr{B} \to \mathbf{Q} \oplus ((\mathscr{A}_1 \otimes \bar{\mathscr{B}}) \otimes \mathscr{A}') = \mathscr{B}_1$ which is a quasi-isomorphism gives an algebra splitting $\mathscr{B}_1 = \mathscr{B} \oplus W$. The elements in W are sums of monomials $a \otimes b \otimes c$ with $a \in \mathscr{A}$, $b \in \bar{\mathscr{B}}$, $c \in \mathscr{A}'$ at least one of a or c not being a constant. This shows that the splitting is also differential because for such a monomial we have $d(a \otimes b \otimes c) \in W$. It follows that the projection $\mathscr{B} \oplus W \to \mathscr{B}$ is a dga morphism. Let $u : \mathscr{B}_1 \to \mathscr{B}$ be this projection. Clearly, it will also be a quasi-isomorphism. Define $g''' = u \circ g''$. We obtain a commutative diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "g"] \arrow[d, "s"'] & \mathscr{B} \arrow[d, "h_1"] \\
R \arrow[r, "g''"] \arrow[d, "\psi_2"'] \arrow[dr, "g'''"'] & \mathscr{B}_1 \arrow[d, "u"] \\
\mathscr{A}_1 & \mathscr{B}
\end{tikzcd}$$

Here $\psi_2 = \psi_1 \circ j$, $j \circ s = i_2'$, $\psi_2 \circ s = f$, s is a quasi-isomorphism and $u \circ h_1 = id$.

LEMMA 3.10. *There is an acyclic ideal $J \subset R$ such that g''' and ψ_2 vanish on J and* $(\overline{R/J})^{(n+2)} = 0$.

This result implies immediately the factorization of both g''' and ψ_2 through $\mathscr{A}_2 = R/J$.

Proof of lemma. Consider the short exact sequence:

$$0 \to (\mathscr{A}_1 \otimes \bar{\mathscr{A}}) \otimes_d \mathscr{A}' \to (\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}' \to \mathscr{A}_1 \otimes_d \mathscr{A}' \to 0$$

Let $F_p(\mathscr{A}_1) = \bar{\mathscr{A}}_1^p$, $F_q(\mathscr{A}) = \bar{\mathscr{A}}^q$, $F_n((\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}') = \sum_{p+q=n} F_p(\mathscr{A}_1) \otimes F_q(\mathscr{A}) \otimes \mathscr{A}'$, $F_n(\mathscr{A}_1 \otimes_d \mathscr{A}') = F_n(\mathscr{A}_1) \otimes \mathscr{A}'$ and $F_n(R) = F_n((\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}') \cap R$. All of these are differential filtrations. They induce spectral sequences and at the E_0 term we get a short exact sequence:

$$0 \to E_0(R) \to E_0(\mathscr{A}_1) \otimes E_0(\mathscr{A}) \otimes \mathscr{A}' \xrightarrow{h} E_0(\mathscr{A}_1) \otimes \mathscr{A}' \to 0$$

The internal differential in $\mathscr{A}'$ is null. It follows that, when we pass to E_1, the map $H^*(h)$ will remain surjective. Hence at the E_1 term we get a short exact sequence:

$$0 \to E_1(R) \to E_1(\mathscr{A}_1) \otimes E_1(\mathscr{A}) \otimes \mathscr{A}' \xrightarrow{H^*(h)} E_1(\mathscr{A}_1) \otimes \mathscr{A}' \to 0$$

Now $E_1(\mathscr{A}) = E_1(\mathscr{A}_0)$ (recall $\mathscr{A} = \mathscr{A}_0 \otimes \mathscr{F}$ with $\mathscr{A}_0$ minimal and $\mathscr{F}$ acyclic). Notice that $E_2^{>n,*}(\mathscr{A}_1 \otimes_d \mathscr{A}') = 0$. Also $E_2^{>n,*}((\mathscr{A}_1 \otimes \mathscr{A}) \otimes_d \mathscr{A}') = 0$. This last fact follows because we have an obvious isomorphism: $(E_1(\mathscr{A}_1), d_1) \otimes \mathscr{H} \simeq ((E_1(\mathscr{A}_1) \otimes E_1(\mathscr{A}_0)) \otimes_d \mathscr{A}', d_1)$ with $\mathscr{H}$ free acyclic, $\mathscr{H} = (\mathscr{A}_0 \otimes_d \mathscr{A}', d_1)$ (as algebras $E_1(\mathscr{A}_0) = \mathscr{A}_0$; the isomorphism corresponds to just a change of generators).

The induced long exact sequence of the E_2 terms shows $E_2^{>n+1,*}(R) = 0$. Let $I = F_{n+2}(R)$ and let $\{u_i\}$ be a minimal set of elements of $F_{n+1}(R)$ such that:

1. The u_i's survive at the E_1^{n+1} term of the spectral sequence giving the classes $[u_i]$.
2. $\{d_1([u_i])\}$ spans the image of d_1 in E_1^{n+2}.

In this case $J = I + \sum \mathbf{Q} u_i$ is an acyclic ideal of R. Moreover $(\overline{R/J})^{n+2} = 0$ and both ψ_2 and g''' are null on J.

Remark. The spectral sequence argument used above is analogous to one used by Felix and Halperin in their study of the Ganea spaces. By a similar method it is possible to show that if $F \to X \to Y$ is a fibration of finite type, simply-connected spaces, then $nil(X/F) \leq nilX + 1$. This strongly suggests $CatX = nilX$.

COROLLARY 3.11. *If* $\Sigma Z \xrightarrow{f} X \to Y$ *is a cofibration sequence of finite type, simply-connected, rational spaces, then* $\mathrm{nil}\,Y \leq \mathrm{nil}X + 1$.

Obviously this implies $nil \leq Cl$.

Proof. Because ΣZ is a formal space we may represent f by a map $f': \mathcal{M}(X) \to H = H^*(\Sigma Z)$. In a standard way we may find a free dga $\mathcal{M}'$, quasi-isomorphic to $\mathcal{M}(X)$ and a surjective map $f'': \mathcal{M}' \to H$ which represents also f. By applying the previous proposition we get a commutative diagram:

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "f''"] \arrow[d, "h"'] & H \arrow[d, "id"] \\ \mathcal{A} \arrow[r, "g"] & H \end{tikzcd}$$

where $\mathcal{A}$ is simply connected of finite type, h is a quasi-isomorphism and $\bar{\mathcal{A}}^{(n+2)} = 0$; $\mathbf{Q} \oplus Ker(g)$ represents Y and it is nilpotent of the right order.

Remark. The ideal proof for the inequality $nil \leq Cl$ would consist in an extension of Lemma 3.2 to the case of Lie algebras with differential filtrations of arbitrary length. It is not clear if such a generalization is true.

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Department of Mathematics
University of Rochester
Rochester, NY 14627
U.S.A.